Achieving a Fair Matching under Partial Information

Ryoji Wada[1] , **Kentaro Yahiro**[1] , **Taiki Todo**[1,2] and **Makoto Yokoo**[1,2]
[1]Kyushu University, Fukuoka, Japan
[2]RIKEN, Center for Advanced Intelligence Project AIP, Tokyo, Japan
{r-wada, yahiro}@agent.inf.kyushu-u.ac.jp, {todo, yokoo}@inf.kyushu-u.ac.jp

Abstract

Two-sided matching with partial information is an extended two-sided matching model, where students have *partial information* on their preferences over the set of schools, and vice versa. The student-proposing Lazy Gale-Shapley algorithm (LGS), proposed for one-to-one matching problem with partial information by Rastegari et al. (2013), minimizes the number of interviews when the partial preferences that schools have over the set of students are identical. The main aim of this paper is to investigate to what extent LGS ensures the minimality of the number of interviews. We provide a sufficient condition on schools' preferences to guarantee that LGS minimizes the number of interviews for many-to-one matching. We then experimentally show that the number of interviews under LGS is in average much smaller than that under a naïve Gale-Shapley implementation where all the required interviews are performed a priori.

1 Introduction

Two-sided matching is one of the traditional and fundamental problems in the economic theory [Gale and Shapley, 1962], where each agent in one side, say, a student, is matched to an agent in the other side, say, a school. One of the best studied variants is *many-to-one* matching, where an agent in one side can match to more than one agent in the other side. Many-to-one matching has a lot of practical applications, such as school choice [Kurata *et al.*, 2017; Hamada *et al.*, 2017] and hospital-residency matching [Kamada and Kojima, 2015].

In traditional matching problems, each agent is assumed to have strict preferences over the agents in the other side. However, such an assumption on the preciseness of preferences does not always hold in practice. For instance, in school choice, each student may have insufficient information to precisely evaluate each school. *Matching problems with partial information* capture and formalize such notion. Agents are endowed with partial knowledge of their preferences, but can refine them and achieve their own underlying strict preferences, e.g., through (costly) interviews [Lee and Schwarz, 2009; Rastegari *et al.*, 2013].

The canonical Gale-Shapley (GS) algorithm returns a student-optimal matching, where no pair of agents would form a blocking pair and no other stable matching Pareto dominates the current one. The *Lazy Gale-Shapley* policy (in short, LGS) is an extension of GS for one-to-one matching with partial information [Rastegari *et al.*, 2013]. By performing interviews, LGS learns the agents' underlying preferences and returns the student-optimal matching, with respect to the underlying preferences, as GS does for the case of complete information, i.e., all the agents completely know their underlying true preferences. Furthermore, under some natural assumptions, LGS minimizes the number of interviews among all policies that return a stable matching whenever all the schools have the identical partial preference.

However, LGS is defined only for one-to-one matching, and as far as the authors know, there have been no discussion on extending LGS for many-to-one matching. Furthermore, even though LGS is guaranteed to minimize the number of interviews, few literatures address its average performance. Therefore, we extend LGS so that many-to-one matching can also be handled, as well as deeply analyze its performance.

We first define the extended implementation of LGS for many-to-one matching problem. We assume that each school may have unacceptable students, to which it prefers unmatched. Although this assumption is in a complementary fashion with partial preferences, in practice it is quite natural; you may only have partial knowledge of your own preference, but at the same time you completely know the candidates that you never accept. Under the assumption, we provide a sufficient condition on schools' preferences to guarantee that LGS minimizes the number of interviews. The condition only applies to schools that cannot accept all of its acceptable students, and requires that there is a partial preference over students, with which each school's partial preference *only on its acceptable students* coincides. Actually, it is a generalization of the "identical partial preferences" condition by Rastegari et al. (2013) for the case of one-to-one matching.

We then compare the performance of LGS with a naïve implementation of GS for many-to-one matching with partial information, in which all the required interviews are performed a priori. Our simulations reveal that, LGS reduces more interviews when preferences are more precise, and the similarity of the preferences of students (schools) does not affect much on the performance.

2 Model

In this paper, we study many-to-one (two-sided) matching problem with partial information. An instance of many-to-one matching problem with partial information is given as a tuple $(S, C, p_S, \succ_S, p_C, \succ_C, q_C)$.

- $S = \{s_1, \ldots, s_n\}$ is the set of n students.
- $C = \{c_1, \ldots, c_m\}$ is the set of m schools.
- $p_S = (p_s)_{s \in S}$ is a profile of partial preferences of students, where each p_s is the partial preference of student s. More specifically, p_s partitions $C \cup \{\varnothing\}$ into finite equivalence classes $(p_s^1, p_s^2, \ldots)$, where the symbol $\varnothing$ indicates that the student is not assigned to any school. Each $p_s^i \subseteq C \cup \{\varnothing\}$, $\bigcup_i p_s^i = C \cup \{\varnothing\}$, and for any $i \neq j$, p_s^i and p_s^j are disjoint.
- $p_C = (p_c)_{c \in C}$ is a profile of partial preferences of schools, where each p_c is the partial preference of school c over $S \cup \{\varnothing\}$, where $\varnothing$ here indicates that the school receives no student. More specifically, p_c partitions $C \cup \{\varnothing\}$ into finite equivalence classes $(p_c^1, p_c^2, \ldots)$, which is defined analogously to p_s.
- $\succ_S = (\succ_s)_{s \in S}$ is a profile of underlying preferences of students, where each $\succ_s$ is the underlying strict preference of student s over $C \cup \{\varnothing\}$, which must be consistent with p_s. We call $\succ_s$ consistent with p_s (and denote $\succ_s \lhd p_s$) if for any $c \in p_s^i$ and $c' \in p_s^j$ such that $i < j$ holds, $c \succ_s c'$ holds.
- $\succ_C$ is a profile of underlying preferences of schools, where each $\succ_c$ is the underlying strict preference of school c over $S \cup \{\varnothing\}$, which must be consistent with p_c. The consistency is defined analogously to p_s and $\succ_s$. We denote $\succ_c \lhd p_c$ if $\succ_c$ is consistent with p_c.
- $q_C = (q_c)_{c \in C} \in \mathbb{N}^m_{>0}$ is a profile of quotas (capacity limit) of schools.

Let us introduce several concepts. School c is acceptable for student s if $c \succ_s \varnothing$ holds. Student s is acceptable for school c if $s \succ_c \varnothing$ holds. A matching μ is an assignment between students and schools, such that each student is assigned to at most one school. $\mu(s) \in C \cup \{\varnothing\}$ denotes the school where s is matched, and $\mu(c) \subseteq S$ denotes the set of students assigned to c. We assume $\mu(s) = c$ if and only if $s \in \mu(c)$ holds. $\mu(s) = \varnothing$ means s is not assigned to any school. A matching μ is student-feasible if for each $s \in S$, either $\mu(s) = \varnothing$ or $\mu(s) \succ_s \varnothing$ holds. A matching μ is school-feasible if for each $c \in C$, $|\mu(c)| \leq q_c$ and for each $s \in \mu(c)$, $s \succ_c \varnothing$ holds. A matching is feasible if it is student and school feasible.

Definition 1 (Stability). *Under a matching μ, a pair (s, c) is a blocking pair if $s \notin \mu(c)$, $c \succ_s \mu(s)$, and either (i) $|\mu(c)| < q_c$ or (ii) there exists $s' \in \mu(c)$ such that $s \succ_c s'$ holds. A matching is stable if it has no blocking pair.*

Definition 2 (Student Optimality). *A matching is* student optimal *if it is stable and weakly preferred by all students to any other stable matching.*

It is guaranteed that there exists a unique student-optimal matching when agents have strict preferences [Gale and Shapley, 1962]. To achieve the student-optimal matching, we need to get more accurate information on preferences of both students and schools. One way is to perform *interviews*, proposed by Rastegari et al. (2013). Here we formally define the process of interviews. An interview between a student s and a school c is represented as $(s : c)$. Through an interview, a student s completely understand how he likes the school c, and can compare with the other schools that he already interviewed. We assume each student $s \in S$ (or $c \in C$) can compare c (or s) and $\varnothing$ if $(s : c)$ takes place.

An information state represents a part of underlying true preference revealed by a sequence of performed interviews.

Definition 3 (Information State). *The information state $\mathcal{I}_s$ of a student $s \in S$ is the strict ordering of the interviewed schools. The global information state $\mathcal{I}_S$ is given as $\bigcup_{s \in S} \mathcal{I}_s$. We analogously define the information states $\mathcal{I}_c$ and the global information state $\mathcal{I}_C$ for schools.*

We denote $c \in \mathcal{I}_s$ (resp. $s \in \mathcal{I}_c$) if interview $(s : c)$ has already been performed. An information state $\mathcal{I}_s$ (resp. $\mathcal{I}_c$) *refines* a partial preference p_s if $\mathcal{I}_s$ is consistent with p_s, i.e., $\mathcal{I}_s \lhd p_s$ holds.

Definition 4 (Policy). *A policy is a procedure, which performs a sequence of interviews and returns a matching for given S, C, p_S, p_C, and q_C. A policy is* sound *if it is guaranteed to return a student-optimal matching under the underlying true preference $(\succ_S, \succ_C)$.*

Definition 5 (Diligence). *A policy is* diligent *if it is sound and for the obtained matching μ, $\mu(s) = c$ holds only if the interview between s and c is performed.*

Definition 6 (Very Weak Dominance). *A policy f* very weakly dominates *another sound policy g if f performs no more interviews than g for any underlying preference profile. A policy* very weakly dominants *if it is sound and very weakly dominates any other sound policy.*

3 Lazy Gale-Shapley for Many-to-One Matching

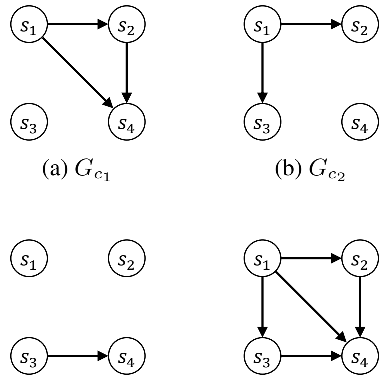
The Lazy Gale-Shapley (LGS), proposed by Rastegari et al. (2013), is a policy based on GS for two-sided matching with complete information, i.e., all the agents have complete knowledge of their own preferences. We extend LGS for many-to-one matching, which is formally described below.

Definition 7 (Lazy Gale-Shapley for Many-to-One Matching). *We assume that, for each $s \in S$, her partial preference, i.e., the sequence of equivalence classes $(p_s^1, p_s^2, \ldots)$, is truncated such that it does not contain any equivalence class that is strictly worse than $\varnothing$. Given a partial ordering $o = (o^1, o^2, \ldots)$ of students, LGS runs as follows:*

Init.: *Set μ to an empty assignment, and for each $s \in S$, l_s to 0, and set $\mathcal{I}_s$ such that it only contains $\varnothing$. For each $c \in C$, set $\mathcal{I}_c$ such that it only contains $\varnothing$, set k to 1.*

Stage k (≥ 1): *For each $s \in o^k$ do the followings:*

Step 1: *Set l_s to $l_s + 1$. If $p_s^{l_s}$ exists in p_s, do the followings. Otherwise, go to Step 2.*



(a) G_{c_1} (b) G_{c_2}

(c) G_{c_3} (d) G_C

Figure 1: Example of compatible preferences

1. *Perform an interview $(s : c)$ for each $c \in p_s^{l_s}$. Update $\mathcal{I}_s$ and $\mathcal{I}_c$ accordingly.*
2. *If s strictly prefer $\varnothing$ to c in $\mathcal{I}_s$, remove c from $\mathcal{I}_s$.*
3. *If c strictly prefer $\varnothing$ to s in $\mathcal{I}_c$, remove s from $\mathcal{I}_c$.*
4. *If $\varnothing$ is the best in $\mathcal{I}_s$, repeat Step 1.*

Step 2: *If $\varnothing$ is the best in $\mathcal{I}_s$, set $\mu(s) \leftarrow \varnothing$. Otherwise, choose school c_t that is ranked the best in $\mathcal{I}_s$ and set $\mu(s) \leftarrow c_t$ and $\mu(c_t) \leftarrow \mu(c_t) \cup \{s\}$.*

Step 3: *If $q_{c_t} < |\mu(c_t)|$ holds, the school c_t rejects student $s_w \in \mu(c_t)$ who is ranked worst in $\mathcal{I}_{c_t}$, i.e., set $\mu(s_w) \leftarrow \varnothing$ and $\mu(c_t) \leftarrow \mu(c_t) \setminus \{s_w\}$.*

Step 4: *If $q_{c_t} = |\mu(c_t)|$ holds, choose student $s_w \in \mu(c_t)$ who is ranked worst in $\mathcal{I}_{c_t}$. Then, for each $s'_w \in S$ who is ranked strictly worse than s_w in either p_{c_t} or $\mathcal{I}_{c_t}$, remove c_t from both $p_{s'_w}$ and $\mathcal{I}_{s'_w}$.*

If some student s' is rejected in Step 3 above, set $s \leftarrow s'$ and go to Step 1. If the assignment for every $s \in S$ is fixed, return μ and terminate. If the assignment for every $s \in o^1 \cup \cdots \cup o^k$ is fixed, go to Stage $k+1$.

3.1 Properties of Lazy Gale-Shapley

To discuss the properties of LGS in detail, let us first introduce an *achievability*. Given a matching μ and school c such that $|\mu(c)| = q_c$, let $w_c \in S$ be the student who is ranked worst by c in μ. A school c is *achievable* by student s under μ if (i) s is acceptable by c, and either of the following holds: (ii-a) $|\mu(c)| < q_c$, or (ii-b) $|\mu(c)| = q_c$ and s is not ranked strictly worse than w_c in p_c. When school c is not achievable by student s, the application of s for c is refused by c.

Theorem 1. *LGS returns a student-optimal matching and runs in polynomial-time for many-to-one matching.*

Rastegari et al. (2013) showed that, for one-to-one matching, LGS minimizes the number of interviews, among all diligent policies, when schools' partial preferences are identical.

Theorem 2 (Rastegari et al. 2013)**.** *LGS is a very weakly dominant, diligent policy in the one-to-one setting where schools are endowed with identical partial preferences, i.e., $p_c = p_{c'}$ for any $c, c' \in C$.*

We then provide a sufficient condition to guarantee the minimality for many-to-one matching. To present our main theorem, we first define some additional notations. Given partial preference p_c of school c, consider a corresponding digraph $G_c = (V, A_c)$ such that $V := S$ and for any pair $s, s'(\neq s) \in S$, $(s, s') \in A_c$ if and only if s is strictly preferred to s' under p_c. Let $S_c := \{s \in S \mid \varnothing \text{ is not strictly preferred to } s \text{ under } p_c\}$ be the set of students who is acceptable (i.e., not worse than receiving no student) for school c under *partial* preference p_c. Given q_C and $S_C = (S_c)_{c \in C}$, let $C_+ \subseteq C$ be the set of schools whose quota is strictly less than the number of acceptable students, i.e., $C_+ := \{c \in C \mid q_c < |S_c|\}$. Given p_C and q_C, let $G_C = (V, A_C)$ be another digraph, which is defined as an edge-union of G_c for all $c \in C_+$, i.e., $V := S$, and $A_C := \bigcup_{c \in C_+} A_c$. Finally, given digraph $G = (V, A)$ and any subset $W \subseteq V$ of vertices, let $G[W]$ be the subgraph induced in G by W.

Definition 8 (Compatible Preferences)**.** *A profile p_C of partial preferences is* compatible *if for any $c \in C_+$,*

$$G_c[S_c] = G_C[S_c].$$

The following example shows a compatible profile of partial preferences.

Example 1. *Consider the case where there are four students, $S = \{s_1, s_2, s_3, s_4\}$, and three schools, $C = \{c_1, c_2, c_3\}$, where quotas for the schools are set as $q_{c_1} = q_{c_2} = 2$, and $q_{c_3} = 1$. Partial preferences of schools are given as follows:*

$$\begin{aligned} p_{c_1} &: (\{s_1\}, \{s_2\}, \{s_4, \varnothing\}, \{s_3\}) \\ p_{c_2} &: (\{s_1\}, \{s_2, s_3, \varnothing\}, \{s_4\}) \\ p_{c_3} &: (\{s_3\}, \{s_4, \varnothing\}, \{s_1, s_2\}) \end{aligned}$$

Note that $S_{c_1} = \{s_1, s_2, s_4\}$, $S_{c_2} = \{s_1, s_2, s_3\}$, and $S_{c_3} = \{s_3, s_4\}$. Fig. 1 describes the corresponding digraphs, G_{c_1}, G_{c_2}, and G_{c_3}, as well as the edge-union graph G_C. The readers can see that $G_{c_1}[S_{c_1}] = G_C[S_{c_1}]$, $G_{c_2}[S_{c_2}] = G_C[S_{c_2}]$, and $G_{c_3}[S_{c_3}] = G_C[S_{c_3}]$, which shows that the profile $(p_{c_1}, p_{c_2}, p_{c_3})$ of partial preferences is compatible.

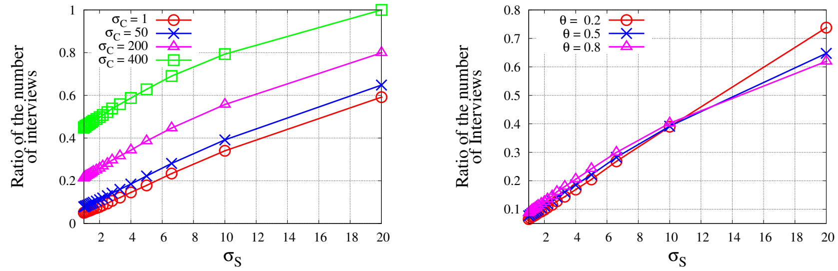
Obviously, when schools' partial preferences are identical, the profile is also compatible. Also, when a profile is compatible, it must be the case that the digraph G_C is acyclic; otherwise $G_c[S_c] = G_C[S_c]$ never holds, since each G_c is transitive, and therefore acyclic.

Given a compatible profile p_C, we first construct a partial ordering o as follows: Put all the students/vertices in G_C with indegree of zero in the first equivalence class o^1, and remove them from G_C, with their outgoing edges. Then, put all the remaining vertices with indegree of zero in the second equivalence class o^2, and remove them with their outgoing edges, and so on. We then run LGS with o obtained in this manner, which guarantees the minimality of the number of interviews.

Theorem 3. *LGS is a very weakly dominant, diligent policy for many-to-one matching when the profile p_C of schools' partial preferences is compatible.*

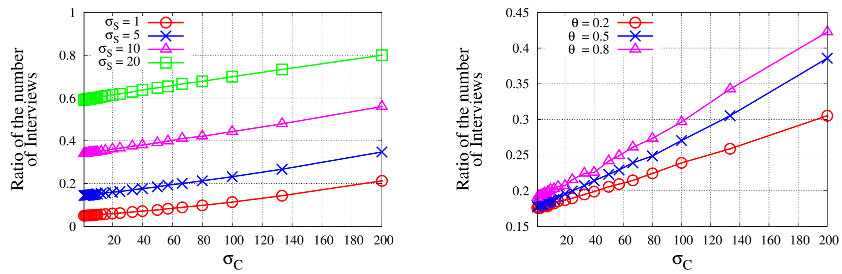
4 Evaluation of #Reduced Interviews

In this section, we evaluate the performance of LGS for many-to-one matching by computer experiments. More precisely, we compare the number of interviews under LGS with



(a) Effect of σ_C ($\theta = 0.5$) (b) Effect of θ ($\sigma_C = 50$)

Figure 2: The ratio of the number of interviews; varying σ_S



(a) Effect of σ_S ($\theta = 0.5$) (b) Effect of θ ($\sigma_S = 5$)

Figure 3: The ratio of the number of interviews; varying σ_C

that under a naïve implementation of GS by running all the interviews required to reveal all the underlying preferences, i.e., $n \cdot m$ interviews, a priori. We assume that all the schools have the same partial preferences. We also assume that each student is acceptable for each school, and vice versa.

We set $n = 400$ and $m = 20$, and a plot in each graph shows an average ratio of the number of interviews to $n \cdot m\,(= 8000)$ over 100 instances. All school's capacities are 20. Since LGS is diligent, the minimum number of interviews for an instance is 400, and the ratio never goes below 0.05.

Students' Preferences For each student s, we first produce an underlying strict preference $\succ_s$ based on the Mallows model [Tubbs, 1992; Lu and Boutilier, 2014; Drummond and Boutilier, 2013] with spread parameter θ, and then, choose the number of equivalence classes d, so that each student's partial preference p_s can be produced by randomly dividing the list $\succ_s$ into d pieces. Let σ_S denote an average size of each student's equivalence classes in p_s, i.e., $\sigma_S = m/d$.

Schools' Preferences We first randomly choose a strict order of all the students, and given number e, split the order into e pieces, which is set as the common partial preferences of schools, i.e., p_c is the same for each school. Let σ_C denote an average size of each school's equivalence classes in p_c, i.e., $\sigma_C = n/e$. Each school c is then assigned a strict underlying preference $\succ_c$ that is obtained by randomly permuting the students in each equivalence class of given p_c.

In Figs. 2 (a) and (b), the x-axis corresponds to the parameter σ_S, and the y-axis corresponds to the average ratio of the number of interviews in LGS to the 8000 interviews mentioned above. In Fig. 2 (a), θ is set to 0.5, and each curve corresponds to a different value of σ_C. For any value of σ_C, the ratio drastically increases as σ_S becomes larger. This results also imply that more interviews tend to be required when schools have little information. In Fig. 2 (b), the value of σ_C is set to 50, and each curve corresponds to a different value of θ. The readers can see that the value of θ has little effect on the number of interviews.

In Figs. 3 (a) and (b), the x-axis corresponds to σ_C, and the y-axis corresponds to the average ratio as well. In Fig. 3 (a), the value of θ is set to 0.5, and each curve corresponds to a different value of σ_S. The ratio gradually increases as σ_C becomes larger, i.e., schools have less information. In Fig. 3 (b), the value of parameter σ_S is set to 5, and each curve corresponds to a different value of θ. Similarly to Fig. 2 (b), more interviews are needed as σ_C increases, even for any θ.

5 Concluding Remarks

We extended LGS for the many-to-one matching problem and provided a sufficient condition on schools' preferences to guarantee the minimality of the number of interviews in LGS, which is a generalization of the "identical equivalence class" condition [Rastegari *et al.*, 2013]. Providing a complete characterization, i.e., a necessary and sufficient condition on schools' preferences, to guarantee the minimality of the number of interviews in LGS is an obvious future work.

Acknowledgments

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